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I. Solution by **G. B. M. ZERR, A.M., Ph.D.**, Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

$$(y+1)^5-y^5=2201.$$

$$(y^2+y)^2+y^2+y=420.$$

$$\therefore\ y^2+y=20 \text{ or } -21.$$

$$y=4 \text{ or } -5 \text{ or } -\tfrac{1}{2}(1\mp\sqrt{-83}).$$

$$x=5 \text{ or } -4 \text{ or } \tfrac{1}{2}(1\pm\sqrt{-83}).$$

II. Solution by the **PROPOSER.**

$5=n$, $\triangle=1$. $\therefore$ Table (I), below, $1^5+5xy.1^3+5x^2y^2\ (1)=2101$; so therefore $5[x^2y^2.1^1+xy.1^3]=2101-1^5$.

$\therefore\ \frac{1}{5}[2101-1^5]=420$; hence $x^2y^2+xy+\frac{1}{4}=420+\frac{1}{4}$......(3), as $\triangle=1$, here.

$\therefore\ xy=-0.5\pm20.5$, and $4xy=80$ or -84.

$$\begin{array}{rl} (2)\text{......}(x-y)^2= & \underline{\ \ 1\ \ } \qquad \underline{\ \ \ 1\ \ } \\ \therefore\ (x+y)^2= & 81 \text{ or } -83. \end{array}$$

$\frac{1}{2}$(the root of (4) + (2) = $\quad x=\frac{1}{2}[1\pm9]$ or $0.5[1\pm i\sqrt{83}$......(5).

$\frac{1}{2}$(the root of (4) − (2) = $\quad y=\frac{1}{2}[-1\pm9]$ or $0.5[-1\pm i\sqrt{83}$, the eight roots.

That is, $x=5$ to $y=4$; $x=-4$ to $y=-5$; etc., as at (5).

Power Difference Theorems. Given $\triangle=x-y=b$, and any one of $w=xy$, or x^n-y^n, or x^m+y^m, $=a$, to find all by *quadratic* to $n=5$; *cubic* to $n=9$; *quintic* to $n=11$, etc. n=odd, viz., 3...5...7...9...11, etc.; m=even integers.

$$\text{(I)}.\ x^n-y^n=\Delta^n+nw\Delta^{n-2}+[\tfrac{1}{2}(n-1)(n-2)-(n-2)^0]w^2\Delta^{n-4}+$$
$$n[\tfrac{1}{6}(n-1)(n-2)-(n-3)]w^3\Delta^{n-6}+n[\tfrac{1}{24}(n-6)^3-(n-6)]w^4\Delta^{n-8}+$$
$$n[\tfrac{1}{120}(n-6)\{(n-8)^3-(n-8)\}]w^5\Delta^{n-10}+\text{......}$$
$$+\tfrac{1}{24}(n^3-n)w^{\frac{1}{2}(n-3)}\Delta^3+nw^{\frac{1}{2}(n-1)}\Delta.$$

$$\text{(II)}.\ x^m+y^m=\Delta^m+mw\Delta^{m-2}+[\tfrac{1}{2}(m-1)(m-2)-(m-2)^0]w^2\Delta^{m-4}+\text{......}$$
$$(\text{like } n)\text{......}+\tfrac{1}{4}(m^2)w^{\frac{1}{2}(m-2)}\Delta^2+2w^{\frac{1}{2}m}.$$

Also solved by *H. C. WHITAKER.*

131. Proposed by **HARRY S. VANDIVER**, Bala, Pa.

It is well known that, when we define the symbol $\sqrt[n]{a}$ after the manner of elementary text-books on algebra, certain *irrational equations* may be written down which have no real or imaginary roots. Required then, the condition, if any, between a, b, c, and d such that the equation, $ax+b+\sqrt{(cx^2+d)}=0$, shall have no root, real or imaginary.

I. Solution by **G. B. M. ZERR, A. M., Ph.D.**, Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$$(a^2-c)x^2+2abx=d-b^2, \text{ or } (c-a^2)x^2-2abx=b^2-d.$$

$$\therefore x=\frac{1}{c-a^2}[ab\pm\sqrt{(a^2d+b^2c-cd)}].$$

If $c>a^2$ there is no root satisfying the equation.

If $c=a^2$, $x=\dfrac{d-b^2}{2ab}$.

Then if $d>b^2$ no root will satisfy the equation.

Whenever x is positive no root can be found.

II. Solution by the **PROPOSER.**

In connection with this problem it will be well to review the subject of irrationals somewhat.

The symbol $\sqrt[n]{a}$ where a is a positive rational number will be defined as the positive rational number satisfying the relation $x^n=a$. Making use of the theory of binomial equations, then the other roots of $x^n=a$, can be represented by $\omega_n\sqrt[n]{a}$, $\omega_n^2\sqrt[n]{a}$......$\omega_n^{n-1}\sqrt[n]{a}$, ω_n being a primitive root of $x^n=1$.

From this $\sqrt[n]{(a+bi)}$ can be defined, a and b being rational and $i^2=-1$ for $a+bi=\rho\text{cis}\theta$ ($\rho\text{cis}\theta=\rho(\cos\theta+i\sin\theta)$) of the Argand diagram, and $\sqrt[n]{(a+bi)}=\sqrt[n]{(\rho\text{cis}\theta)}\pm\text{cis}\theta/n\sqrt[n]{\rho}$, which is intelligible since

$$\sqrt[n]{\rho}=\sqrt[n]{[\sqrt{(a^2+b^2)}]}=\sqrt[2n]{(a^2+b^2)},$$

which symbol has been defined.

The above definitions treat the irrational symbol as one-valued and agree with those given in most of the recent text-books on algebra. If we define the radical symbol as multi-valued, the irrational equation loses all its individuality and might just as well be termed a rational equation.

It is a well known fact that there are radical equations which possess no roots real or imaginary, but a thorough classification of such equations has never been given, so far as I am aware. In the course of the following solution this matter will be cleared up.

Consider the simplest possible irrational equation:

$$\sqrt{x}=1......(1).$$

This possesses one root $x=1$, but the related equation, $\sqrt{x}=-1$......(2), does not possess a root, which may be rigorously demonstrated as follows.

The equation $(\sqrt{x}+1)(\sqrt{x}-1)=0$......(3), must contain all the roots of (1) and (2) for a value of x which reduces either factor to zero, reduces the product to zero. Expanding (3) there is obtained

$$x-1=0,\text{ or }x=1,$$

which satisfies (1) but not (2). Hence $\sqrt{x}+1=0$ possesses no root, and is the simplest irrational equation of this type.

Proceeding to the case of the equations

$$ax+b+\sqrt{(cx^2+d)}=0\ldots\ldots(4),$$
$$ax+b-\sqrt{(cx^2+d)}=0.,\ldots.(5).$$

Put $ax+b=y$, then (4) becomes

$$-\sqrt{c\left(\frac{y-b}{a}\right)^2+d}=+y\ldots\ldots(6),$$

and (5) becomes

$$\sqrt{c\left(\frac{y-b}{a}\right)^2+d}=+y\ldots\ldots(7).$$

If (6) is to be satisfied by a value of y, this value cannot be positive since $\sqrt{[f(y)]}+1=0$ is an impossible relation as was proved in the case of (2). The product of (6) and (7) becomes, after simplifying,

$$(c-a^2)y^2+2bcy+a^2d+b^2c=0\ldots\ldots(8).$$

Now it is evident that if (8) gives a positive value for y then (6) is not satisfied but (7) is; if (8) gives a negative root the reverse is the case.

In this way the following criteria are deduced, assuming that a^2d+b^2c-cd is positive. Putting

$$A=\frac{2bc}{c-a^2}, \text{ and } B=\frac{a^2d+b^2c}{c-a^2}.$$

If $A>0$ and $B>0$, then (4) has no roots and (5) has two roots.

If $A>0$ and $B<0$, or $A<0$ and $B<0$, then (4) has one root and (5) has one root.

If $A<0$ and $B>0$, then (4) has two roots and (5) has no roots..

This solves the problem as originally stated. In an analogous way one could deduce criteria for the solubility of irrational equations of any type whatever.

This problem was also discussed by J. M. Boorman, who arrived at the conclusion that there is no condition for the equation to have no roots. He maintains that Newton's (his) theorem "Every algebraic equation has as many roots as it has dimensions," holds good. But this is not true. D'Alembert's theorem does not apply in this case at all since the equation is not a rational algebraic equation. It may not be out of place to state here D'Alembert's theorem in full: "Every rational, integral, algebreic equation whose coefficients are imaginary quantities, or, in special cases, real quantities, has at least one root." When this theorem is proved it follows as a corollary that there are as many roots as there are units in the number expressing the degree of the equation. Ed.

132.. Proposed by J. SCHEFFER. A. M., Hagerstown, Md.

Solve $2^x+3^y=4$; $5^x+6^y=7$.